LECTURE 4

THE BASIC CONVEX SETS

IN THIS TALK,

WE'LL TAKE A

CLOSER LOOK AT

THE BASIC

CONVEX SETS

$T_\ell(x, M)$

INTRODUCED IN

THE LAST TALK.

WE WILL INTRODUCE AN ADDITIONAL FAMILY OF CONVEX SETS

$\sigma_\ell(x)$

CLOSELY RELATED TO $\Gamma_\ell(x, M)$.

WE WILL EXHIBIT

THE BASIC PROPERTIES

OF THE Γ_ℓ AND σ_ℓ,

AND WE WILL DEFINE

OTHER FAMILIES

OF CONVEX SETS THAT

SHARE THOSE

BASIC PROPERTIES.

The study of the

Γ_ℓ AND σ_ℓ

is FUNDAMENTAL

in proving all the

main results in

these lectures.

IN THIS TALK, WE'LL GET ACQUAINTED WITH Γ'S & σ'S.

IN THE NEXT TALK, WE'LL APPLY WHAT WE'VE LEARNED, TO PROVE ...

THE MAIN THM FROM LECTURE 3:

$$\Gamma_{\ell_*}(x, M) \subset \Gamma(x, CM).$$

RECALL: IN LECTURE 3, WE REDUCED ALL OUR RESULTS ON C^m INTERPOLATION OF $f: E \to \mathbb{R}$ $(\#E < \infty)$ TO THE ABOVE MAIN THM.

GENERALIZE THE PREVIOUS SETTING

- Fix $m, n \geq 1$.
- We work in $C^m(\mathbb{R}^n)$.
- $J_x(F)$ DENOTES THE $(m-1)^{\text{rst}}$ DEGREE TAYLOR POLY OF F AT x
- $\mathcal{P}$ = VECTOR SPACE OF ALL SUCH POLYS

GIVEN:

$E \subset \mathbb{R}^n$ finite, $f: E \to \mathbb{R}$, $M > 0$

So far, we've tried to find

$F \in C^m(\mathbb{R}^n)$ that agrees

EXACTLY with f.

WE NOW

BROADEN

THE DISCUSSION

BY INTRODUCING

A

"TOLERANCE"

$\varepsilon : E \to \mathbb{R}$.

WE ARE GIVEN E, f, ε, M.

WE TRY TO FIND A FUNCTION

$F \in C^m(\mathbb{R}^n)$

such that

$$\boxed{\begin{array}{l} \|F\|_{C^m(\mathbb{R}^n)} \le M \text{ and} \\ |F(x) - f(x)| \le M\,\varepsilon(x) \quad \forall x \in E. \end{array}}$$

(OUR PREVIOUS INTERPOLATION PROBLEM IS THE CASE $\varepsilon \equiv 0$.)

THE "TRUE" Γ's & σ's

FIX M, E, f, ε AS ABOVE.

LET $x \in \mathbb{R}^n$. (Maybe $x \in E$, maybe not)

DEFINE

$$\boxed{\Gamma(x, M) = \{ J_x(F) : F \in C^m, \|F\|_{C^m} \le M, \; |F(x) - f(x)| \le M\varepsilon(x) \; \forall x \in E \}}$$

Taking $M = 1$, $f \equiv 0$, WE ARRIVE AT

$$\boxed{\sigma(x) = \{ J_x(G) : G \in C^m, \|G\| \le 1, \; |G(x)| \le \varepsilon(x) \; \forall x \in E \}.}$$

So $\Gamma(x, M)$ is a (possibly empty) convex subset of $\mathcal{P}$, determined by x, E, f, ε, M,

while

$\sigma(x) \subset \mathcal{P}$ is a SYMMETRIC convex SET $(P \in \sigma(x) \Leftrightarrow -P \in \sigma(x))$ determined by x, E, ε.

The sets $\Gamma(x, M)$, $\sigma(x)$

CARRY A LOT OF INFORMATION

ABOUT INTERPOLATION.

For instance, by definition, the interpolation problem

$$\boxed{\begin{array}{l} |F(x)-f(x)| \leq M\,\varepsilon(x) \quad (\text{all } x \in E) \\ \|F\|_{C^m(\mathbb{R}^n)} \leq M \end{array}}$$

has a solution if & only if

$\Gamma(x, M) \neq \emptyset$ (any x).

RECALL, OUR INTERPOLATION PROBLEM IS DELICATE IF E LIES NEAR THE ZERO SET OF ONE OR SEVERAL POLY'S

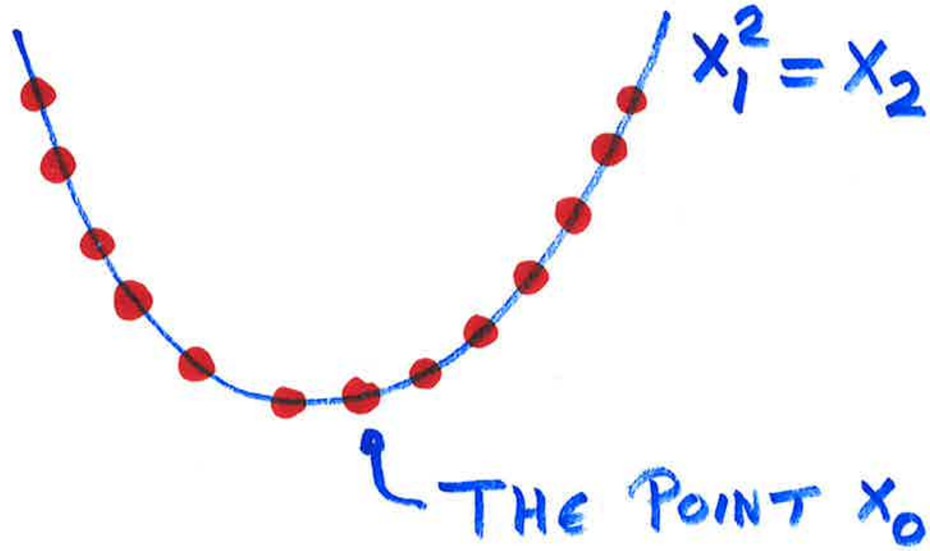
$P_1, \dots, P_L$.

THE CONVEX SET $\sigma(x)$ ENCODES THOSE POLYS.

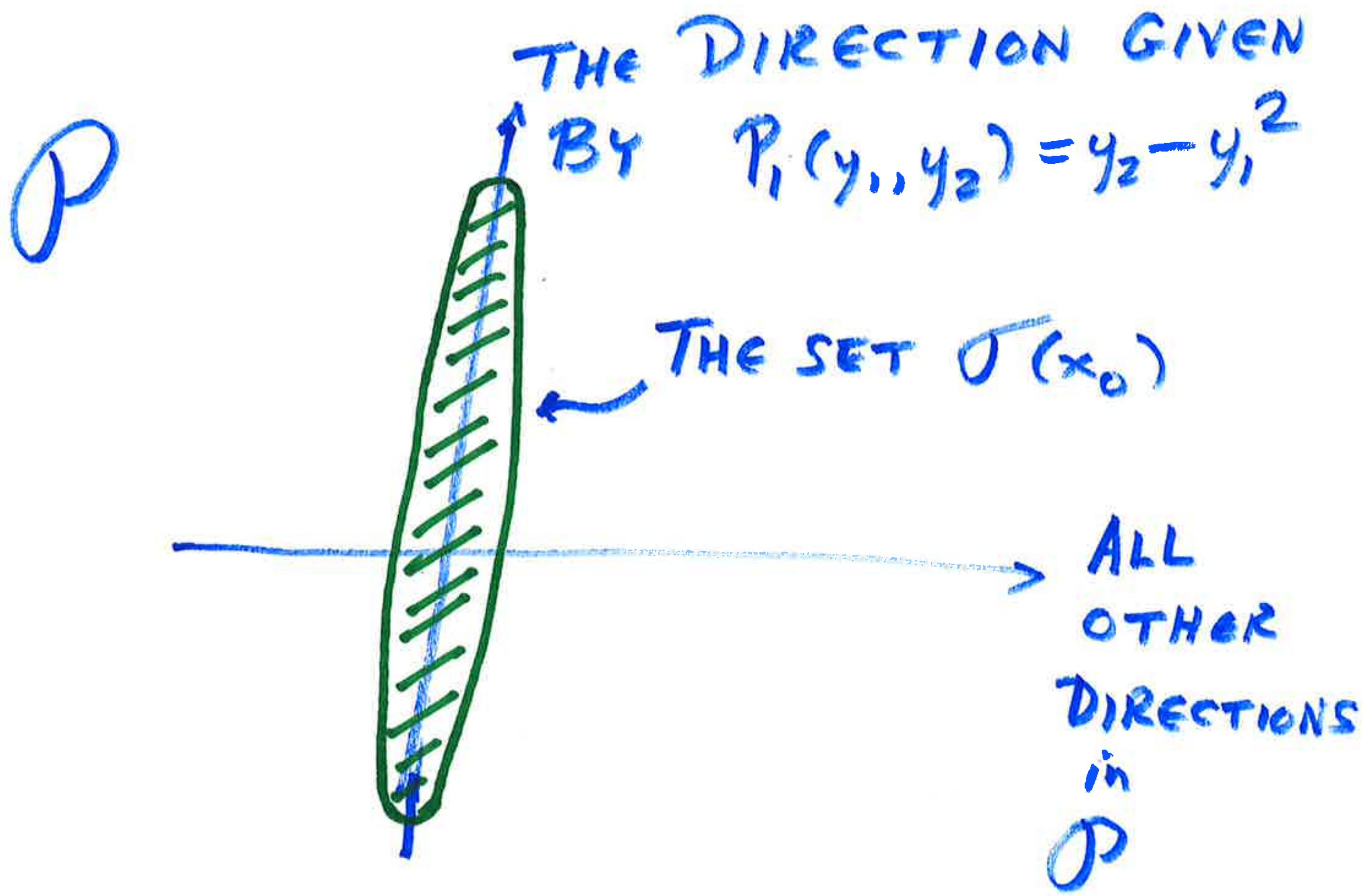
LET'S SEE AN EXAMPLE.

An Example in $C^3(\mathbb{R}^2)$

The Set E



The Set $\sigma(x_0)$



MORAL

The polys whose zero sets are the DIRECTIONS in the VECTOR SPACE $\mathcal{P}$ IN WHICH $\sigma(x)$ is

LONG

may make trouble

BACK TO THE GENERAL CASE

WE WANT TO COMPUTE THE APPROXIMATE SIZE & SHAPE OF

$$\Gamma(x,M) = \{ J_x(F) : \|F\| \leq M,\ |F - f| \leq M\varepsilon \text{ on } E \}$$

and

$$\sigma(x) = \{ J_x(G) : \|G\| \leq 1,\ |G| \leq \varepsilon \text{ on } E \}$$

AS IN LECTURE 3,

WE WILL DEFINE CONVEX SETS

$\Gamma_\ell(x, M)$, $\sigma_\ell(x)$ $\begin{pmatrix} \ell = 0, 1, \dots \\ x \in E \end{pmatrix}$

BY INDUCTION ON ℓ.

OUR MAIN THM WILL ASSERT

THT

$$\Gamma(x, M) \subset \Gamma_{\ell_*}(x, M) \subset \Gamma(x, CM)$$

for $x \in E$, WHERE ℓ_* & C

depend only on m, n (in $C^m(\mathbb{R}^n)$).

OUR Γ_ℓ & σ_ℓ WILL SATISFY:

$$\Gamma_\ell(x, M) \supset \Gamma_{\ell+1}(x, M) \supset \Gamma(x, M)$$

and

$$\sigma_\ell(x) \supset \sigma_{\ell+1}(x) \supset \sigma(x).$$

THE INDUCTION ON ℓ

BASE CASE $\ell = 0$.

$$\boxed{\begin{aligned} &\Gamma_0(x, M) = \\ &\{ P \in \mathcal{P} : |P(x) - f(x)| \leq M \, \varepsilon(x) \\ &\qquad \text{and } |\partial^\alpha P(x)| \leq M \text{ for } |\alpha| \leq m-1 \} \end{aligned}}$$

$$\boxed{\begin{aligned} \sigma_0(x) = \{ Q \in \mathcal{P} : &\ |Q(x)| \leq \varepsilon(x) \\ &\text{and } |\partial^\alpha Q(x)| \leq 1 \text{ for } |\alpha| \leq m-1 \} \end{aligned}}$$

$(x \in E)$

INDUCTION STEP.

Fix $\ell \geq 0$, $M > 0$. Suppose we have defined $\Gamma_\ell(x, M)$, $\sigma_\ell(x)$ for all $x \in E$.

We will define $\Gamma_{\ell+1}(x, M)$, $\sigma_{\ell+1}(x)$ for all $x \in E$.

$\Gamma_{\ell+1}(x, M)$ CONSISTS OF ALL $P \in \Gamma_\ell(x, M)$

SUCH THAT FOR EACH $y \in E$

THERE EXISTS $P' \in \Gamma_\ell(y, M)$

SUCH THAT

$|\partial^\alpha(P - P')(x)| \leq M|x-y|^{m-|\alpha|}$ for $|\alpha| \leq m-1$.

$\sigma_{\ell+1}(x)$ consists of all $Q \in \sigma_\ell(x)$

such that for each $y \in E$

there exists $Q' \in \sigma_\ell(y)$ such that

$|\partial^\alpha(Q - Q')(x)| \leq |x-y|^{m-|\alpha|}$ for $|\alpha| \leq m-1$.

COMPARED TO THE Γ_ℓ DEFINED

IN LECTURE 3 (FOR THE CASE $\mathcal{E} \equiv 0$)

OUR Γ_0 HAS BEEN MODIFIED,

BUT THE INDUCTION STEP

(PASSING FROM THE Γ_ℓ TO THE $\Gamma_{\ell+1}$)

IS UNCHANGED.

THE σ_ℓ ARE ANALOGOUS TO THE Γ_ℓ.

AS IN LECTURE 3, AN EASY INDUCTION ON ℓ USING TAYLOR'S THM. SHOWS THAT

$\Gamma_\ell(x, M) \subset \mathcal{P}$ IS A (POSSIBLY EMPTY) CONVEX SET

$\sigma_\ell(x) \subset \mathcal{P}$ IS A SYMMETRIC CONVEX SET

$$\Gamma_\ell(x, M) \supset \Gamma(x, M)$$

$$\sigma_\ell(x) \supset \sigma(x)$$

$$\Gamma_\ell(x, M) \supset \Gamma_{\ell+1}(x, M)$$

$$\sigma_\ell(x) \supset \sigma_{\ell+1}(x).$$

OUR GOAL IS TO PROVE THAT

$$\Gamma_{l_*}(x, M) \subset \Gamma(x, CM)$$

WITH l_*, C DEPENDING ONLY ON m, n IN $C^m(\mathbb{R}^n)$.

THAT WILL GENERALIZE the MAIN THM ON THE Γ's FROM LECTURE 3, because we no longer assume that $\varepsilon \equiv 0$.

To prepare the way, we derive the basic properties of our Γ's and σ's

WE LOOK FIRST AT THE "TRUE" Γ and σ.

$\boxed{\text{Let } P_1 \in \Gamma(x, M_1) \text{ and } P_2 \in \Gamma(x, M_2). \text{ Then } P_1 - P_2 \in (M_1 + M_2)\,\sigma(x).}$

$\boxed{\text{Conversely, if } P \in \Gamma(x, M_1) \text{ and } Q \in M_2\,\sigma(x), \text{ then } P + Q \in \Gamma(x, M_1 + M_2).}$

(THESE ASSERTIONS ARE IMMEDIATE FROM THE DEFINITIONS OF Γ & σ.)

AS A CONSEQUENCE, WE HAVE THE FOLLOWING INCLUSIONS.

Suppose $P \in \Gamma(x, M)$.

Then

$$P + M\sigma(x) \subset \Gamma(x, 2M) \subset P + 3M\sigma(x).$$

ROUGHLY SPEAKING,

if $\Gamma(x, M)$ is non-empty,

then $\Gamma(x, M)$ IS ESSENTIALLY A TRANSLATE

OF $M\sigma(x)$.

So $\sigma(x)$ TELLS US HOW MUCH

"ROOM TO MANEUVER" THERE IS

IN $\Gamma(x, M)$.

WE HAVE WORKED WITH

THE "TRUE" Γ and σ,

BUT ANALOGOUS PROPERTIES

HOLD ALSO FOR Γ_ℓ and σ_ℓ,

by an EASY INDUCTION on ℓ.

WE ARE NOW READY TO DESCRIBE

THE KEY PROPERTIES OF THE

Γ_ℓ , σ_ℓ .

WE HAVE SEEN ALMOST ALL OF THEM ALREADY.

THE KEY PROPERTIES OF Γ_ℓ & σ_ℓ

$\Gamma_\ell(x, M) \subset \mathcal{P}$

IS A (POSSIBLY EMPTY) CONVEX SET.

(0)$_\Gamma$ If $M \leq M'$, then $\Gamma_\ell(x, M) \subset \Gamma_\ell(x, M')$.

Any $P \in \Gamma_\ell(x, M)$ satisfies

$|\partial^\alpha P(x)| \leq M$ for $|\alpha| \leq m-1$

$\sigma_\ell(x) \subset \mathcal{P}$ IS A SYMMETRIC CONVEX SET.

$(0)_\sigma$ Any $Q \in \sigma_\ell(x)$ satisfies

$|\partial^\alpha Q(x)| \le 1$ for $|\alpha| \le m-1$.

(1) If $P \in \Gamma_\ell(x, M)$, then

$$\boxed{P + M\sigma_\ell(x) \subset \Gamma_\ell(x, C_1 M) \subset P + C_2 M \sigma_\ell(x).}$$

Properties $(0)_\Gamma$ & $(0)_\sigma$

are obvious,

and we have just discussed

Property (1).

$(2)_\Gamma$ $\boxed{\begin{array}{l}\text{Let } \ell \geq 1, \text{ and let } P \in \Gamma_\ell(x, M). \\ \text{Given any } y \in E, \text{ there exists} \\ P' \in \Gamma_{\ell-1}(y, M) \text{ such that} \\ |\partial^\alpha (P - P')(x)| \leq M |x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m-1.\end{array}}$

$(2)_\sigma$ $\boxed{\begin{array}{l}\text{Let } \ell \geq 1, \text{ and let } Q \in \sigma_\ell(x). \\ \text{Given any } y \in E, \text{ there exists} \\ Q' \in \sigma_{\ell-1}(y) \text{ such that} \\ |\partial^\alpha (Q - Q')(x)| \leq |x-y|^{m-|\alpha|} \text{ for } |\alpha| \leq m-1.\end{array}}$

These assertions simply restate the Induction Step in the Definition of Γ_ℓ & σ_ℓ.

There's one more crucial property of the σ_ℓ.

(3)$_\sigma$ Let $P \in \sigma_\ell(x)$, $Q \in \mathcal{P}$, $0 < \delta \le 1$.

Suppose P and Q satisfy:

$$|\partial^\alpha P(x)| \le \delta^{m-|\alpha|} \quad \text{for } |\alpha| \le m-1$$

and

$$|\partial^\alpha Q(x)| \le \delta^{-|\alpha|} \quad \text{for } |\alpha| \le m-1.$$

Then $J_x(PQ) \in C_W\, \sigma_\ell(x)$,

where C_W depends only on m, n, ℓ.

WE SAY THAT $\sigma_\ell(x)$ IS "WHITNEY ℓ-CONVEX" (at x), WITH "WHITNEY CONSTANT" C_W.

The assumptions on P & Q

in $(3)_\sigma$

look familiar from

<u>Whitney's classical proof</u>,

in which one writes

$$F = F_{\hat{Q}} + \sum_Q \theta_Q \cdot (F_Q - F_{\hat{Q}}).$$

There, the estimates we assumed for P

will hold for $F_Q - F_{\hat{Q}}$,

and the estimates we assumed for Q

will hold for θ_Q.

Let us prove $(3)_\sigma$ for the "TRUE SIGMA", $\sigma(x)$.

The proof for the $\sigma_\ell(x)$ proceeds by induction on ℓ using similar ideas.

We may as well suppose $x = 0$.

Let $P \in \sigma(0)$, $Q \in \mathcal{P}$, $0 < \delta \leq 1$.

Assume the estimates

$$\left[\begin{array}{l} |\partial^\alpha P(0)| \leq \delta^{m-|\alpha|} \\ |\partial^\alpha Q(0)| \leq \delta^{-|\alpha|} \end{array}\right] \text{ for } |\alpha| \leq m-1$$

Because $P \in \sigma(0)$, there exists $G \in C^m(\mathbb{R}^n)$

with

$\|G\|_{C^m} \leq 1$, $|G(x)| \leq \varepsilon(x)$ (all $x \in E$), $J_0(G) = P$.

LET θ BE A SMOOTH CUTOFF FN. ON $\mathbb{R}^n$,

SATISFYING:

$$\boxed{\begin{array}{l} \theta = 1 \text{ IN A NEIGHBORHOOD OF } 0 \\[2em] \theta(x) = 0 \text{ FOR } |x| \geq \delta \\[2em] |\partial^\alpha \theta(x)| \leq C\delta^{-|\alpha|} \text{ FOR } |\alpha| \leq m,\ x \in \mathbb{R}^n. \end{array}}$$

OUR ESTIMATES FOR P, Q, G, θ

IMPLY EASILY THAT THE FUNCTION

$$\tilde{G} = \theta P Q$$

SATISFIES

$$\boxed{\begin{gathered} \|\tilde{G}\|_{C^m} \leq C, \quad |G(x)| \leq C\varepsilon(x) \;(\text{all } x \in E); \\ \text{and} \\ J_0(\tilde{G}) = J_0(PQ). \end{gathered}}$$

Therefore, $J_0(PQ) \in C\sigma(0)$,

AS ASSERTED IN $(3)_\sigma$.

LET'S BELIEVE $(3)_\sigma$ for the $\sigma_\ell(x)$,

AS STATED.

Finally, we recall that $\Gamma_\ell(x, M) \supset \Gamma(x, M)$, which means that the following holds:

(!) Let $F \in C^m(\mathbb{R}^n)$, $M > 0$.

Suppose that $\|F\|_{C^m(\mathbb{R}^n)} \leq M$

and that $|F(x) - f(x)| \leq M\varepsilon(x)$ (all $x \in E$).

Then $J_x(F) \in \Gamma_\ell(x, M)$ for all $x \in E$, $\ell \geq 0$.

THE BASIC PROPERTIES

OF THE

Γ_ℓ AND σ_ℓ

ARE

$(0)_\Gamma$, $(0)_\sigma$, (1), $(2)_\Gamma$, $(2)_\sigma$, $(3)_\sigma$,

and

(!).

RECALL THAT CONSTANTS C_1, C_2 ENTER INTO (1), AND THE CONSTANT C_w ENTERS INTO $(3)_\sigma$.

These are the ONLY properties we will use regarding the Γ_ℓ & σ_ℓ.

We are ready to state our MAIN RESULT.

MAIN THM:

For a large enough ℓ_*, depending only on m, n, the following holds.

Let Γ_ℓ, σ_ℓ satisfy $(0)_\Gamma$, $(0)_\sigma$, (1), $(2)_\Gamma$, $(2)_\sigma$, $(3)_\sigma$, with constants C_1, C_2, C_W.

$$\boxed{\begin{array}{l} \text{Given } P_0 \in \Gamma_{\ell_*}(x_0, M_0), \\ \text{there exists } F \in C^m(\mathbb{R}^n) \text{ such that} \\ \bullet\ \|F\|_{C^m(\mathbb{R}^n)} \le C M_0 \\ \bullet\ J_x(F) \in \Gamma_0(x, CM_0) \text{ for all } x \in E \\ \bullet\ J_{x_0}(F) = P_0 . \end{array}}$$

Here, C depends only on m, n, C_1, C_2, C_W.

COROLLARY: For the Γ's and σ's arising from our interpolation problems with "tolerance" $\varepsilon : E \to [0, \infty)$,

we have

$$\Gamma(x, M) \subset \Gamma_{l_*}(x, M) \subset \Gamma(x, CM)$$

with C depending only on m & n.

WE HAVE SUCCEEDED IN COMPUTING THE APPROXIMATE SIZE & SHAPE OF THE Γ's.

So TODAY'S MAIN THM implies the MAIN THM ON Γ_ℓ from LETURE 3, in a MORE GENERAL FORM (because we aren't restricted to $\varepsilon \equiv 0$).

CONSEQUENTLY,

FROM TODAY'S MAIN THM

WE OBTAIN ALL THE RESULTS

FROM LECTURE 3,

IN A MORE GENERAL FORM

(involving $\mathcal{E} : E \to [0, \infty)$)

FOR THE MOMENT, WE WON'T STATE THOSE GENERALIZATIONS.

INSTEAD, WE EXPLORE

$$\boxed{\text{THE IMPLICATIONS of the fact that today's MAIN THM holds for } \underline{\text{any}} \text{ family of } \Gamma\text{'s} \ \& \ \sigma\text{'s satisfying } (0)_\Gamma \cdots (3)_\sigma.}$$

TO START WITH, WE SKETCH

TWO VERY DIFFERENT

WAYS TO DEFINE

OTHER FAMILIES

OF Γ_ℓ, σ_ℓ SATISFYING $(0)_\rho \cdots (3)_\sigma$,

AND EXPLAIN

WHY WE CARE.

For the FIRST ALTERNATIVE

FAMILY OF Γ_ℓ, σ_ℓ,

WE RETURN TO THE

$$\boxed{\text{BRUDNYI-SHVARTSMAN FINITENESS PRINCIPLE.}}$$

FINITENESS THM: Given $m, n \geq 1$ there exist $k^\# = k^\#(m,n)$ and $C^\# = C^\#(m,n)$ for which the following holds.

Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite. Let $M > 0$.

Suppose that for every subset $S \subset E$ with $\#(S) \leq k^\#$ there exists $F^S \in C^m(\mathbb{R}^n)$ with norm $\leq M$ such that $F^S = f$ on S.

Then there exists $F \in C^m(\mathbb{R}^n)$ with norm $\leq C^\# M$, such that $F = f$ on E.

SO, TO UNDERSTAND WHETHER AN INTERPOLANT EXISTS, IT IS ENOUGH TO EXAMINE ALL "SMALL SUBSETS" $S \subset E$, i.e., SUBSETS WITH $\leq k^{\#}$ POINTS.

IN FACT, WE HAVE ONLY

TO EXAMINE $O(\#E)$

PARTICULAR SUBSETS

$S_1, \dots, S_L \subset E$, EACH

WITH AT MOST $k^\#$ POINTS,

BUT LET'S NOT WORRY

ABOUT THAT NOW.

To prove the

BRUDNYI - SHVARTSMAN

FINITENESS THM,

WE DEFINE CONVEX SETS

Γ_ℓ, σ_ℓ satisfying $(0)_\Gamma \cdots (3)_\sigma$

WITH $\mathcal{E} = 0$.

(COULD ALSO HANDLE GENERAL $\mathcal{E}: E \to [0,\infty)$.

LET'S NOT WORRY ABOUT THAT.)

A NEW FAMILY OF Γ_ℓ, σ_ℓ

GIVEN $x \in E$, $M > 0$, $S \subset E$,

WE DEFINE

$$\boxed{\Gamma(x, M, S) = \left\{ J_x(F) : F \in C^m, \|F\| \le M, F = f \text{ on } S \right\}}$$

SIMILARLY, DEFINE

$$\boxed{\sigma(x, S) = \left\{ J_x(G) : G \in C^m, \|G\| \le 1, G = 0 \text{ on } S \right\}}$$

NOTE THAT

$$\Gamma(x, M, S) \supset \Gamma(x, M, S')$$

and

$$\sigma(x, S) \supset \sigma(x, S')$$

IF $S \subset S'$

LET $D = \dim \mathcal{P}$.

WE DEFINE

$$\boxed{\Gamma_\ell(x, M) = \bigcap_{\substack{S \subset E \\ \#(S) \leq (D+1)^\ell}} \Gamma(x, M, S)}$$

and

$$\boxed{\sigma_\ell(x) = \bigcap_{\substack{S \subset E \\ \#(S) \leq (D+1)^\ell}} \sigma(x, S)}$$

THESE Γ'S & σ'S SATISFY

$(0)_\Gamma \cdots (3)_\sigma$ AND ALSO (!).

MOREOVER, WE HAVE

$$\Gamma_{l_*}(x, M) \neq \phi$$

PROVIDED $k^\#$ IN THE

BRUDNYI-SHVARTSMAN FINITENESS THM

SATISFIES

$$k^\# \geq (D+1)^{l_* + 1}.$$

(WE'LL SEE WHY IN A MOMENT.)

THEREFORE, THE BRUDNYI-SHVARTSMAN FINITENESS THM FOLLOWS AT ONCE FROM TODAY'S MAIN THM, APPLIED TO THE "NEW" T_ℓ & σ_ℓ DEFINED ABOVE.

ACTUALLY, THESE

"NEW"

Γ_ℓ & σ_ℓ

WERE INTRODUCED BEFORE

THE Γ_ℓ & σ_ℓ FROM LECTURE 3.

The DEFINITIONS OF THE

Γ's & σ's, AND THE

PROOF OF $(0)_\Gamma \cdots (3)_\sigma$

ARE BASED ON

HELLY'S THM ON CONVEX SETS:

LET $K_1, \cdots, K_N \subset \mathbb{R}^D$ BE CONVEX.

$(N > D)$.

IF EVERY $(D+1)$ of the K_i

HAVE A POINT IN COMMON,

THEN $K_1 \cap \cdots \cap K_N \neq \emptyset$.

A SHORT, ELEGANT PROOF OF HELLY'S THM, DUE TO RADON, CAN BE FOUND ON WIKIPEDIA by GOOGLING "HELLY'S THEOREM".

LET'S USE HELLY'S THM to check that the hypotheses of the FINITENESS THM imply that

$$\boxed{\begin{array}{l}\Gamma_{\ell_*}(x, M) \neq \emptyset \\ \text{provided} \\ \quad k^\# \geq (D+1)^{\ell_*+1}.\end{array}}$$

By definition, $\Gamma_{\ell_*}(x, M)$

is the intersection of

$\Gamma(x, M, S)$

over all $S \subset E$ with $\#(S) \leq (D+1)^{\ell_*}$.

We want to show that this intersection

is non-empty.

By Helly's thm it is enough to show

that

$\Gamma(x, M, S_1) \cap \cdots \cap \Gamma(x, M, S_{D+1}) \neq \emptyset$

whenever $S_1, \dots, S_{D+1} \subset E$ and $\#(S_i) \leq (D+1)^{\ell_*}$

(all i)

However,

$S := S_1 \cup \cdots \cup S_{D+1} \subset E$ satisfies

$\#(S) \le (D+1)^{\ell_* + 1} \le k^\#.$

The hypothesis of the FINITENESS THM gives an $F^S \in C^m$ WITH NORM $\le M$

S.T. $F^S = f$ on S.

BY DEFINITION, $J_x(F^S) \in \Gamma(x, M, S)$,

So $\Gamma(x, M, S) \ne \emptyset$.

THEREFORE

$\Gamma(x, M, S_1) \cap \cdots \cap \Gamma(x, M, S_{D+1}) \ne \emptyset$,

COMPLETING THE PROOF THAT $\Gamma_{\ell_*}(x, M) \ne \emptyset$,

BECAUSE $\Gamma(x, M, S_i) \supset \Gamma(x, M, S)$.

The same idea, based on Helly's Thm, gives a simple proof of $(2)_\Gamma$ & $(2)_\sigma$ for our "New" Γ_ℓ & σ_ℓ.

We omit the detailed verification of $(0)_\Gamma \cdots (3)_\sigma$ and (!) for our Γ's & σ's.

SO MUCH FOR THE ABOVE

"NEW" Γ's & σ's.

WE TURN OUR ATTENTION TO

$$\boxed{\text{YET ANOTHER FAMILY OF } \Gamma_\ell \text{ \& } \sigma_\ell.}$$

WE WANT TO MAKE

EFFICIENT COMPUTATIONS

TO FIND INTERPOLANTS F

RECALL: To compute an interpolant F

from $f : E \to \mathbb{R}$,

WE PERFORM ONE-TIME WORK,

THEN ANSWER QUERIES.

- A QUERY CONSISTS OF A POINT $x \in \mathbb{R}^n$, and
- THE RESPONSE TO QUERY x IS $J_x(F)$.

SUPPOSE $\#(E) = N$.

WE HAVE PROMISED TO COMPUTE F

USING

ONE-TIME WORK $O(N \log N)$

and

QUERY WORK $O(\log N)$.

TO do SO, WE WILL HAVE TO COMPUTE THE APPROXIMATE SIZE & SHAPE OF THE T_ℓ & σ_ℓ USING $O(N \log N)$ ONE-TIME WORK.

HOW CAN WE DO THAT?

RECALL: OUR ORIGINAL DEF. OF Γ_ℓ

IS BASED ON THE FOLLOWING RULE:

$$P \in \Gamma_{\ell+1}(x, M) \iff$$

for every $y \in E$ there exists $P' \in \Gamma_\ell(y, M)$

such that

$$|\partial^\alpha(P - P')(x)| \leq M\,|x-y|^{m-|\alpha|} \quad (\text{all } \alpha).$$

SO EVERY $y \in E$ TALKS TO EVERY $x \in E$.

THE RELEVANT OBJECTS –

CONVEX SUBSETS OF $\mathcal{P}$ –

MAY BE COMPLICATED.

HOWEVER, EVEN IF CONVEX SETS COULD BE DESCRIBED & MANIPULATED BY JUST ONE COMPUTER OP, THE FACT THAT EVERY y TALKS TO EVERY X IMPLIES THAT IT WILL TAKE AT LEAST $\sim N^2$ OPERATIONS TO COMPUTE THE T_ℓ, σ_ℓ

WE HAVE TO REDUCE THE WORK FROM $\sim N^2$ TO $\sim N \log N$.

HERE, WE'VE USED THE Γ_ℓ, σ_ℓ

FROM LECTURE 3.

TODAY'S "NEW" Γ_ℓ, σ_ℓ,

BASED ON SUBSETS

$S \subset E$ WITH $\#(S) \leq (D+1)^{\ell_*}$,

ARE EVEN WORSE, IN FACT

MUCH, MUCH WORSE!

The number of $S \subset E$

with $\#(S) \leq (D+1)^{\ell_*}$

is of the order of magnitude

$$N^{[(D+1)^{\ell_*}]}$$

UGH!

TO MAKE EFFICIENT COMPUTATIONS,

WE DEFINE

YET ANOTHER FAMILY OF T_ℓ, σ_ℓ,

BASED ON THE

WELL-SEPARATED PAIRS DECOMPOSITION.

The construction of these CHEAP T_ℓ & σ_ℓ isn't so simple. We won't give the definition in these lectures.

Instead, WE SIMPLY NOTE

TWO CRUCIAL POINTS

The Γ_ℓ & σ_ℓ can be computed in $O(N \log N)$ computer operations,

and

The Γ_ℓ and σ_ℓ satisfy $(0)_\Gamma \cdots (3)_\sigma$

and (!)

NEXT WE DISCUSS

A SIGNIFICANT

GENERALIZATION

WE GENERALIZE OUR PREVIOUS DISCUSSION IN TWO RESPECTS

IN PLACE OF $C^m(\mathbb{R}^n)$, WE WILL WORK WITH $C^{m,\omega}(\mathbb{R}^n)$, THE SPACE OF FUNCTIONS WHOSE m^{th} DERIVATIVES HAVE MODULUS OF CONTINUITY ω.

INSTEAD OF ASKING FOR $F \in C^m(\mathbb{R}^n)$

SUCH THAT

$$|F(x) - f(x)| \leq M\,\varepsilon(x) \text{ for all } x \in E,$$

WE ASK FOR $F \in C^{m,\omega}(\mathbb{R}^n)$

SUCH THAT

$$J_x(F) \in P^x + M\,\sigma_0(x) \text{ for all } x \in E.$$

Here, $P^x \in \mathcal{P}$ and

$\sigma_0(x) \subset \mathcal{P}$ is a SYMMETRIC CONVEX SET.

WHY DO WE CARE?

TRUST ME — IT WILL BE ABUNDANTLY CLEAR IN A LATER TALK.

LET'S DISCUSS IN

MORE DETAIL

EACH OF THE CHANGES

- $C^m \rightsquigarrow C^{m,\omega}$

and

- $|F(x) - f(x)| \leq M\varepsilon(x) \rightsquigarrow J_x(F) \in P^x + M\sigma_0(x)$

FIRST, $C^m \rightsquigarrow C^{m,\omega}$

A MODULUS OF CONTINUITY

is a FUNCTION $\omega : [0,\infty) \to [0,1]$

WITH THE FOLLOWING PROPERTIES:

$$\omega(0) = \lim_{t \to 0+} \omega(t) = 0$$

$\omega(t)$ is increasing (MAYBE NOT STRICTLY)

$\omega(t)/t$ is decreasing (MAYBE NOT STRICTLY)

$\omega(t) = 1$ for $t \geq 1$.

EXAMPLE: $\omega(t) = \text{MIN}\,(t^\alpha, 1)$ WITH $0 < \alpha \leq 1$.

LET ω BE A MODULUS OF CONTINUITY.

THEN $C^{m,\omega}(\mathbb{R}^n)$ CONSISTS OF ALL

$F \in C^m(\mathbb{R}^n)$ FOR WHICH THE NORM

$$\boxed{\|F\|_{C^{m,\omega}(\mathbb{R}^n)} := \|F\|_{C^m(\mathbb{R}^n)} + \sup_{\substack{x \neq y \\ |\alpha| = m}} \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{\omega(|x-y|)}}$$

is finite.

If $\omega(t) = \text{MAX}(t^\alpha, 1)$ $(0 < \alpha \leq 1)$

then $C^{m,\omega}(\mathbb{R}^n) = C^{m,\alpha}(\mathbb{R}^n)$.

INTERPOLATION PROBLEMS FOR C^m ARE A SPECIAL CASE OF INTERPOLATION PROBLEMS FOR $C^{m,\omega}$

(<u>PROVIDED E IS FINITE</u> !)

That's because,

given $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ FINITE,

finding an interpolant $F \in C^m(\mathbb{R}^n)$

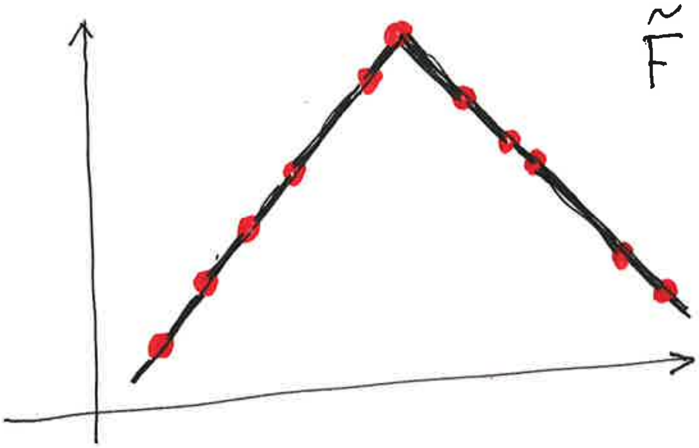
with controlled C^m norm

is equivalent to

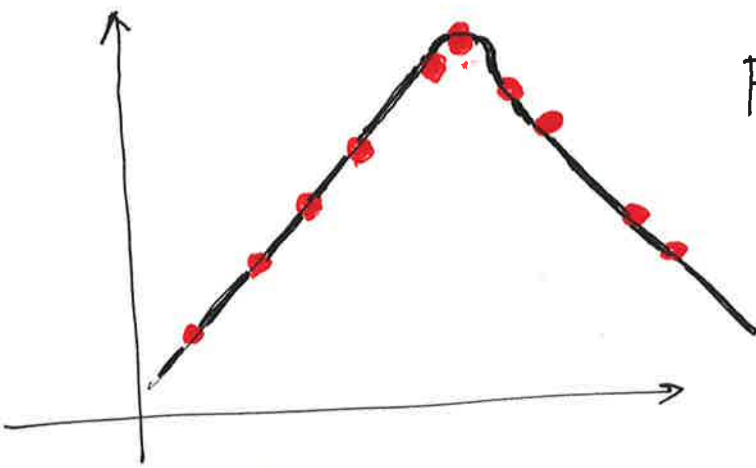
finding an interpolant $\tilde{F} \in C^{m-1,1}(\mathbb{R}^n)$

with controlled $C^{m-1,1}$-norm.

(To obtain F, we SMOOTH OUT $\tilde{F}$ A BIT, THEN MAKE SMALL CORRECTIONS.)



$\tilde{F} \in C^{0,1}(\mathbb{R}^1)$



$F \in C^1(\mathbb{R}^1)$

A SIMPLE 1D EXAMPLE

BECAUSE "C^m INTERPOLATION $\approx$ $C^{m-1,1}$ INTERPOLATION"

WE SHOULD BE DISCUSSING

$m^{\underline{th}}$ DEGREE TAYLOR POLYS

INSTEAD OF $(m-1)^{\underline{RST}}$ DEGREE,

WHEN WE WORK WITH $C^{m,\omega}(\mathbb{R}^n)$.

So for the Rest of this Lecture,

we Define

$$J_x(F) = \left(m^{\underline{th}} \text{ Degree Taylor Poly of } F \text{ at } x\right)$$

and

$$\mathcal{P} = \left(\begin{array}{l}\text{Vector Space of all (Real)} \\ m^{\underline{th}} \text{ Degree Polys on } \mathbb{R}^n\end{array}\right)$$

WHEN WE WORK WITH $C^{m,\omega}(\mathbb{R}^n)$, WE HAVE TO MODIFY THE DEFINITION OF WHITNEY t-CONVEXITY.

LET $\sigma \subset \mathcal{P}$ BE A SYMMETRIC CONVEX SET.

LET $x_0 \in \mathbb{R}^n$.

LET C_w BE A POSITIVE REAL NUMBER

LET ω BE A MODULUS OF CONTINUITY.

WE SAY THAT

σ is WHITNEY ω-CONVEX AT x_0

with Whitney constant C_w if the

following condition holds.

Let $P \in \sigma$, $Q \in \mathcal{P}$, $0 < \delta \leq 1$.

Suppose $|\partial^\alpha P(x_0)| \leq \omega(\delta) \cdot \delta^{m-|\alpha|}$

and $|\partial^\alpha Q(x_0)| \leq \delta^{-|\alpha|}$, for $|\alpha| \leq m$.

Then $J_x(PQ) \in C_w \sigma$.

This agrees with our previous notion of WHITNEY t-CONVEXITY

when we take $\omega(t) = \min(t, 1)$

and we remember that

$C^{m-1,1}$ IS A PROXY FOR C^m

WHEREVER WE USED WHITNEY t-CONVEXITY BEFORE, WE NOW USE WHITNEY ω-CONVEXITY.

So much for the passage from C^m to $C^{m,\omega}$.

We now discuss the passage from interpolation problems involving

$$|F(x) - f(x)| \leq M\varepsilon(x)$$

to interpolation problems involving

$$J_x(F) \in P^x + M\sigma_0(x).$$

THE PROOF OF THE BRUDNYI–SHVARTSMAN FINITENESS THM. BASED ON TODAY'S "NEW" Γ_ℓ & σ_ℓ CAN BE EASILY ADAPTED TO OUR MORE GENERAL SETTING.

WE OBTAIN THE FOLLOWING RESULT.

Generalized Finiteness Thm.

Given $m, n \geq 1$, there exists $k^\# = k^\#(m,n)$ for which the following holds.

Suppose we are given:

- A modulus of continuity ω
- A finite set $E \subset \mathbb{R}^n$
- For each $x \in E$, a poly $P_E^x \in \mathcal{P}$ and a symmetric convex set $\sigma_0(x) \subset \mathcal{P}$
- Positive numbers M, C_w.

WE MAKE THE FOLLOWING

ASSUMPTIONS:

For each $x \in E$, the set $\sigma_0(x)$ is Whitney ω-convex at x, with Whitney constant C_w.

For each subset $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^{m,\omega}(\mathbb{R}^n)$ such that

- $\|F^S\|_{C^{m,\omega}(\mathbb{R}^n)} \leq M$, and
- $J_x(F^S) \in P^x + M\,\sigma_0(x)$ for all $x \in S$

Then there exists $F \in C^{m,\omega}(\mathbb{R}^n)$

such that

$$\boxed{\begin{aligned} &\|F\|_{C^{m,\omega}(\mathbb{R}^n)} \leq CM \\ &\text{and} \\ &J_x(F) \in P^x + CM\,\sigma_0(x) \text{ for } x \in E, \end{aligned}}$$

where C depends only on m, n, C_w.

That's the

GENERALIZED FINITENESS THM.

It holds because today's

MAIN THM (on the Γ_ℓ, σ_ℓ)

requires only that the Γ_ℓ, σ_ℓ

satisfy $(0)_\rho \cdots (3)_\sigma$.

LATER ON, WHEN WE STUDY

C^m EXTENSION OF FUNCTIONS

$f : E \to \mathbb{R}$ WITH $E \subset \mathbb{R}^n$ INFINITE,

THE ABOVE

GENERALIZED FINITENESS THM.

WILL PLAY A CRUCIAL RÔLE.

P. S. To recover the

Brudnyi-Shvartsman Finiteness Thm

from the

Generalized Finiteness Thm,

we take

$P^x =$ the constant poly. $y \mapsto f(x)$

and we set

$$\sigma_0(x) = \{ P \in \mathcal{P} : |\partial^\alpha P(x)| \leq 1, \text{ all } \alpha; \text{ and } |P(x)| \leq \varepsilon(x) \},$$

Note: $\sigma_0(x)$ is Whitney ω-convex at x,

with Whitney constant $C_w = 1$.

RECAP

IN THIS TALK, WE HAVE DESCRIBED SEVERAL FAMILIES OF CONVEX SETS $\Gamma_\ell(x, M)$, $\sigma_\ell(x) \subset \mathcal{P}$, SATISFYING THE KEY PROPERTIES $(0)_\Gamma \cdots (3)_\sigma$.

WE HAVE STATED A <u>MAIN THM.</u> ON FAMILIES Γ_ℓ, σ_ℓ ASSUMED TO SATISFY THOSE PROPERTIES.

WE HAVE SEEN THAT GOOD THINGS FOLLOW FROM THAT MAIN THM.

IN THE NEXT LECTURE,

WE SKETCH THE PROOF

OF TODAY'S

<u>MAIN THM</u> ON T_ℓ, σ_ℓ

(IN THE SPECIAL CASE $\omega(t) \equiv \min(t, 1)$
TO AVOID BORING TECHNICALITIES)

THANK

YOU!